Stress and Strain

13.1 Basic Loading Configurations

An object subjected to an external force will move in the direction of the applied force. The object will deform if its motion is constrained in the direction of the applied force. Deformation implies relative displacement of any two points within the object. The extent of deformation will be dependent upon many factors including the magnitude, direction, and duration of the applied force, the material properties of the object, the geometry of the object, and environmental factors such as heat and humidity.

In general, materials respond differently to different loading configurations. For a given material, there may be different mechanical properties that must be considered while analyzing its response to, for example, tensile loading as compared to loading that may cause bending or torsion. Figure 13.1 is drawn to illustrate different loading conditions, in which an L-shaped beam is subjected to forces $\underline{F}_1$, $\underline{F}_2$, and $\underline{F}_3$. The force $\underline{F}_1$ subjects the arm AB of the beam to tensile loading. The force $\underline{F}_2$ tends to bend the arm AB. The force $\underline{F}_3$ has a bending effect on arm BC and a twisting (torsional) effect on arm AB. Furthermore, all of these forces are subjecting different sections of the beam to shear loading.
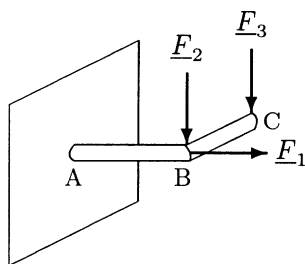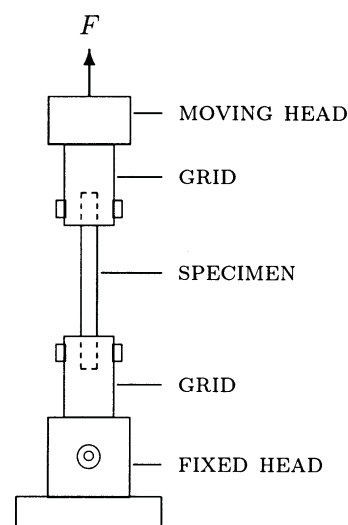
13.2 Uniaxial Tension Test

The mechanical properties of materials are established by subjecting them to various experiments. The mechanical response of materials under tensile loading is analyzed by the *uniaxial* or *simple tension test* that will be discussed next. The response of materials to forces that cause bending and torsion will be reviewed in the following chapter.

The experimental setup for the uniaxial tension test is illustrated in Fig. 13.2. It consists of one fixed and one moving head with attachments to grip the test specimen. A specimen is placed and firmly fixed in the equipment, a tensile force of known magnitude is applied through the moving head, and the corresponding elongation of the specimen is measured. A general understanding of the response of the material to tensile loading is obtained by repeating this test for a number of specimens made of the same material, but with different lengths, cross-sectional areas, and under tensile forces with different magnitudes.

Fig. 13.1 Loading modes

Fig. 13.2 Uniaxial tension test

13.3   Load-Elongation Diagrams

Consider the three bars shown in Fig. 13.3. Assume that these bars are made of the same material. The first and the second bars have the same length but different cross-sectional areas, and the second and third bars have the same cross-sectional area but different lengths. Each of these bars can be subjected to a series of uniaxial tension tests by gradually increasing the applied forces and measuring corresponding increases in their lengths. If F is the magnitude of the applied force and Δl is the increase in length, then the data collected can be plotted to obtain a load versus elongation diagram for each specimen. Effects of geometric parameters (cross-sectional area and length) on the load-bearing ability of the material can be judged by drawing the curves obtained for each specimen on a single graph (Fig. 13.4) and comparing them. At a given force magnitude, the comparison of curves 1 and 2 indicate that the larger the cross-sectional area, the more difficult it is to deform the specimen in a simple tension test, and the comparison of curves 2 and 3 indicates that the longer the specimen, the larger the deformation in tension.
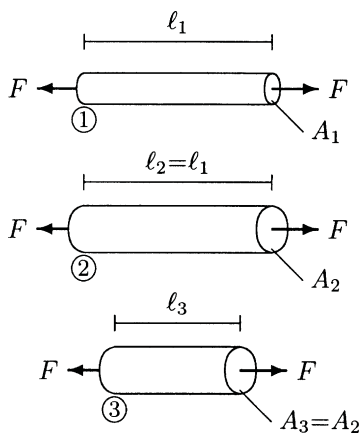
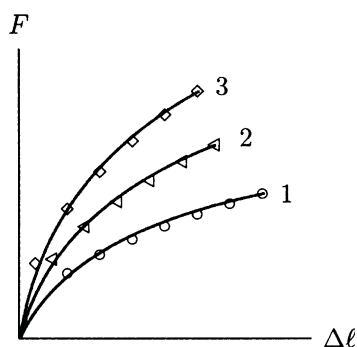Note that instead of applying a series of tensile forces to a single specimen, it is preferable to have a number of specimens with almost identical geometries and apply one force to one specimen only once. As will be discussed later, a force applied on an object may alter its mechanical properties.

Another method of representing the results obtained in a uniaxial tension test is by first dividing the magnitude of the applied force F with the cross-sectional area A of the specimen, normalizing the amount of deformation by dividing the measured elongation with the original length of the specimen, and then plotting the data on a F/A versus $\Delta l/l$ graph as shown in Fig. 13.5. The three curves in Fig. 13.4, representing three specimens made of the same material, are represented by a single curve in Fig. 13.5. It is obvious that some of the information provided in Fig. 13.4 is lost in Fig. 13.5. That is why the representation in Fig. 13.5 is more advantageous than that in Fig. 13.4. The single curve in Fig. 13.5 is unique for a particular material, independent of the geometries of the specimens used during the experiments. This type of representation eliminates geometry as one of the variables, and makes it possible to focus attention on the mechanical properties of different materials. For example, consider the curves in Fig. 13.6, representing the mechanical
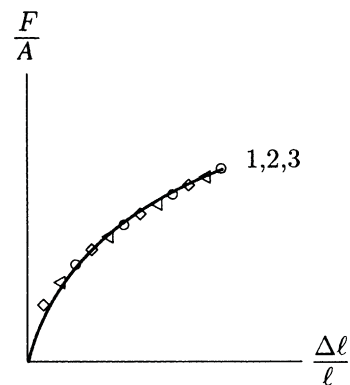

Fig. 13.3 Specimens

Fig. 13.5 Load over area versus elongation over length diagram

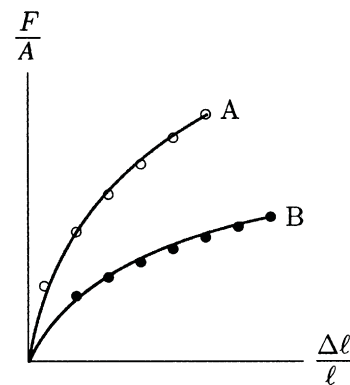

Fig. 13.4 Load-elongation diagrams

Fig. 13.6 Material A is stiffer than material B

behavior of materials A and B in simple tension. It is clear that material B can be deformed more easily than material A in a uniaxial tension test, or material A is "stiffer" than material B.

13.4 Simple Stress

Consider the cantilever beam shown in Fig. 13.7a. The beam has a circular cross-section, cross-sectional area A, welded to the wall at one end, and is subjected to a tensile force with magnitude F at the other end. The bar does not move, so it is in equilibrium. To analyze the forces induced within the beam, the method of sections can be applied by hypothetically cutting the beam into two pieces through a plane ABCD perpendicular to the centerline of the beam. Since the beam as a whole is in equilibrium, the two pieces must individually be in equilibrium as well. This requires the presence of an internal force collinear with the externally applied force at the cut section of each piece. To satisfy the condition of equilibrium, the internal forces must have the same magnitude as the external force (Fig. 13.7b). The internal force at the cut section represents the resultant of a force system distributed over the cross-sectional area of the beam (Fig. 13.7c). The intensity of the internal force over the cut section (force per unit area) is known as the *stress*. For the case shown in Fig. 13.7, since the force resultant at the cut section is perpendicular (normal) to the plane of the cut, the corresponding stress is called a *normal stress*. It is customary to use the symbol σ (sigma) to refer to normal stresses. The intensity of this distributed force may or may not be uniform (constant) throughout the cut section. Assuming that the intensity of the distributed force at the cut section is uniform over the cross-sectional area A, the normal stress can be calculated using:

$$\sigma = \frac{F}{A}. \tag{13.1}$$

If the intensity of the stress distribution over the area is not uniform, then Eq. (13.1) will yield an *average normal stress*. It is customary to refer to normal stresses that are associated with tensile loading as *tensile stresses*. On the other hand, *compressive stresses* are those associated with compressive loading. It is also customary to treat tensile stresses as positive and compressive stresses as negative.

The other form of stress is called *shear stress*, which is a measure of the intensity of internal forces acting parallel or tangent to a plane of cut. To get a sense of shear stresses, hold a stack of paper with both hands such that one hand is under the stack while the other hand is above it. First, press the stack of papers together. Then, slowly slide one hand in the direction parallel to the surface of the papers while sliding the other hand in the opposite direction. This will slide individual papers relative to one another and generate frictional forces on the surfaces of individual papers. The shear stress is comparable to the intensity of the frictional force over the surface area upon which it is applied. Now, consider the cantilever beam illustrated in Fig. 13.8a. A downward force with magnitude F is applied to its free end. To analyze internal forces and moments, the method of sections can be applied by fictitiously cutting the beam through a plane ABCD that is perpendicular to the centerline of the beam. Since the beam as a whole is in equilibrium, the two pieces thus obtained must individually be equilibrium as well. The free-body diagram of the right-hand piece of the beam is illustrated in Fig. 13.8b along with the internal force and moment on the left-hand piece. For the equilibrium of the right-hand piece, there has to be an upward force resultant and an internal moment at the cut surface. Again for the equilibrium of this piece, the internal force must have a magnitude F. This is known as the *internal shearing force* and is the resultant of a distributed load over the cut surface (Fig. 13.8c). The intensity of the shearing force over the cut surface is known as the *shear stress*, and is commonly denoted with the symbol τ (tau). If the area of this surface (in this case, the cross-sectional area of the beam) is A, then:
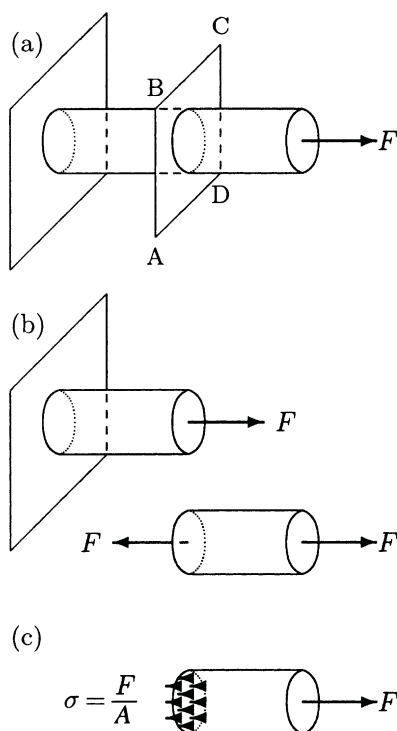
$$\tau = \frac{F}{A}. \tag{13.2}$$
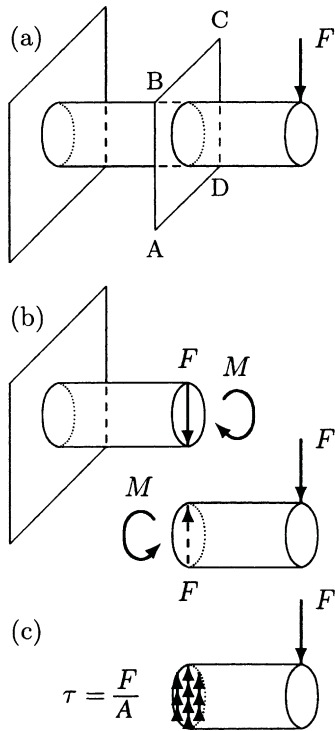

Fig. 13.7 Normal stress

Fig. 13.8 Shear stress

Fig. 13.9 Normal strain

The underlying assumption in Eq. (13.2) is that the shear stress is distributed uniformly over the area. For some cases, this assumption may not be true. In such cases, the shear stress calculated by Eq. (13.2) will represent an average value.

The dimension of stress can be determined by dividing the dimension of force $[F] = [M][L]/[T^2]$ with the dimension of area $[L^2]$. Therefore, stress has the dimension of $[M]/[L][T^2]$. The units of stress in different unit systems are listed in Table 13.1. Note that stress has the same dimension and units as pressure.

Table 13.1 Units of stress

System	Units of stress	Special name
SI	N/m^2	Pascal (Pa)
CGS	dyn/cm^2	
British	lb/ft^2 or lb/in.^{2}	psf or psi

13.5   Simple Strain

Strain, which is also known as unit deformation, is a measure of the degree or intensity of deformation. Consider the bar in Fig. 13.9. Let A and B be two points on the bar located at a distance l_1, and C and D be two other points located at a
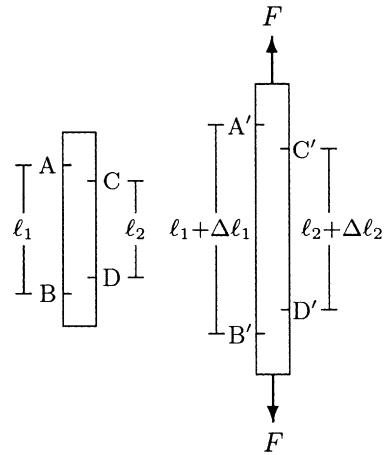
distance l_2 from one another, such that $l_1 > l_2$. l_1 and l_2 are called *gage lengths*. The bar will elongate when it is subjected to tensile loading. Let Δl_1 be the amount of elongation measured between A and B, and Δl_2 be the increase in length between C and D. Δl_1 and Δl_2 are certainly some measures of deformation. However, they depend on the respective gage lengths, such that $\Delta l_1 > \Delta l_2$. On the other hand, if the ratio of the amount of elongation to the gage length is calculated for each case and compared, it would be observed that $(\Delta l_1/\Delta l_1) \simeq (\Delta l_2/\Delta l_2)$. Elongation per unit gage length is known as *strain* and is a more fundamental means of measuring deformation.

As in the case of stress, two types of strains can be distinguished. The *normal* or *axial strain* is associated with axial forces and defined as the ratio of the change (increase or decrease) in length, Δl, to the original gage length, l, and is denoted with the symbol ε (epsilon):

$$\varepsilon = \frac{\Delta l}{l}. \tag{13.3}$$

When a body is subjected to tension, its length increases, and both Δl and ε are positive. The length of a specimen under compression decreases, and both Δl and ε become negative.

The second form of strain is related to distortions caused by shearing forces. Consider the rectangle (ABCD) shown in Fig. 13.10, which is acted upon by a pair of shearing forces. Shear forces deform the rectangle into a parallelogram (AB′C′D). If the relative horizontal displacement of the top and the bottom of the rectangle is d and the height of the rectangle is l, then the *average shear strain* is defined as the ratio of d and l which is equal to the tangent of angle γ (gamma). This angle is usually very small. For small angles, the tangent of the angle is approximately equal to the angle itself. Hence, the average shear strain is equal
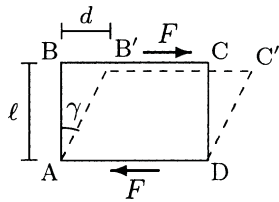
Fig. 13.10 Shear strain

to angle γ (measured in radians), which can be calculated using:

$$\gamma = \frac{d}{l}. \tag{13.4}$$

Strains are calculated by dividing two quantities having the dimension of length. Therefore, they are dimensionless quantities and there is no unit associated with them. For most applications, the deformations and consequently the strains involved are very small, and the precision of the measurements taken is very important. To indicate the type of measurements taken, it is not unusual to attach units such as cm/cm or mm/mm next to a strain value. Strains can also be given in percent. In engineering applications, the strains involved are of the order of magnitude 0.1% or 0.001.

Figure 13.11 is drawn to compare the effects of tensile, compressive, and shear loading. Figure 13.11a shows a square object (in a two-dimensional sense) under no load.
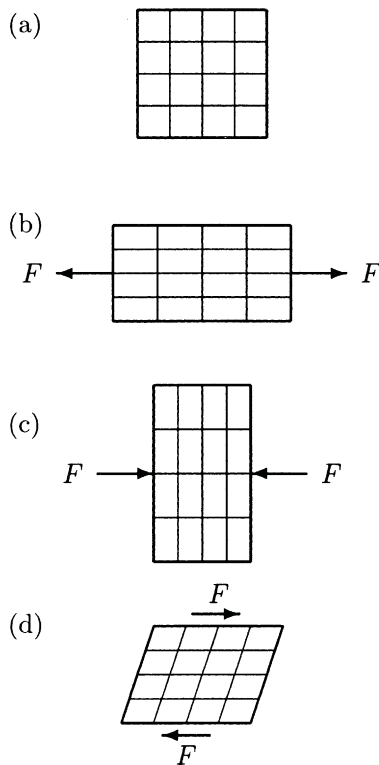
Fig. 13.11 Distortions under tensile (**b**), compressive (**c**), and shear (**d**) loading

The square object is ruled into 16 smaller squares to illustrate different modes of deformation. In Fig. 13.11b, the object is subjected to a pair of tensile forces. Tensile forces distort squares into rectangles such that the dimension of each square in the direction of applied force (axial dimension) increases while its dimension perpendicular to the direction of the applied force (transverse dimension) decreases. In Fig. 13.11c, the object is subjected to a pair of compressive forces that distort squares into rectangles such that the axial dimension of each square decreases while its transverse dimension increases. In Fig. 13.11d, the object is subjected to a pair of shear forces that distort the squares into diamonds.

13.6 Stress–Strain Diagrams

It was demonstrated in Sect. 13.3 that the results of uniaxial tension tests can be used to obtain a unique curve representing the relationship between the applied load and corresponding deformation for a material. This can be achieved by dividing the applied load with the cross-sectional area (F/A) of the specimen, dividing the amount of elongation measured with the gage length ($\Delta l/l$), and plotting a F/A versus $\Delta l/l$ graph. Notice, however, that for a specimen under tension, F/A is the average tensile stress σ and $\Delta l/l$ is the average tensile strain ε. Therefore, the F/A versus $\Delta l/l$ graph of a material is essentially the *stress–strain diagram* of that material.

Different materials demonstrate different stress–strain relationships, and the stress–strain diagrams of two or more materials can be compared to determine which material is relatively stiffer, harder, tougher, more ductile, and/or more brittle. Before explaining these concepts related to the strength of materials, it is appropriate to first analyze a typical stress–strain diagram in detail.

Consider the stress–strain diagram shown in Fig. 13.12. There are six distinct points on the curve that are labeled as O, P, E, Y, U, and R. Point O is the origin of the $\sigma - \varepsilon$ diagram, which corresponds to the initial no load, no deformation stage. Point P represents the *proportionality limit*. Between O and P, stress and strain are linearly proportional, and the $\sigma - \varepsilon$ curve is a straight line. Point E represents the *elastic limit*. The stress corresponding to the elastic limit is the greatest stress that can be applied to the material without causing any permanent deformation within the material. The material will not resume its original size and shape upon unloading if it is subjected to stress levels beyond the elastic limit. Point Y is the *yield point*, and the stress σ_y corresponding to the yield point is called the *yield strength* of the material. At this stress level, considerable elongation (yielding) can occur without a corresponding increase of load. U is the highest stress point on the $\sigma - \varepsilon$ curve. The stress σ_u is the *ultimate strength* of the
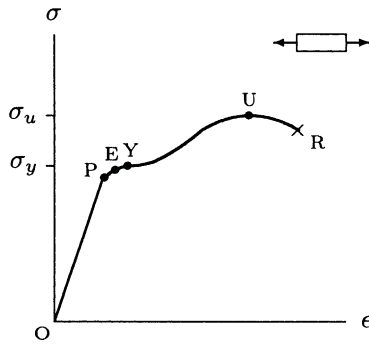
Fig. 13.12 Stress–strain diagram for axial loading

material. For some materials, once the ultimate strength is reached, the applied load can be decreased and continued yielding may be observed. This is due to the phenomena called *necking* that will be discussed later. The last point on the $\sigma - \varepsilon$ curve is R, which represents the *rupture* or *failure point*. The stress at which the rupture occurs is called the *rupture strength* of the material.

For some materials, it may not be easy to determine or distinguish the elastic limit and the yield point. The yield strength of such materials is determined by the *offset method*, illustrated in Fig. 13.13. The offset method is applied by drawing a line parallel to the linear section of the stress–strain diagram and passing through a strain level of about 0.2% (0.002). The intersection of this line with the $\sigma - \varepsilon$ curve is taken to be the yield point, and the stress corresponding to this point is called the *apparent yield strength* of the material.
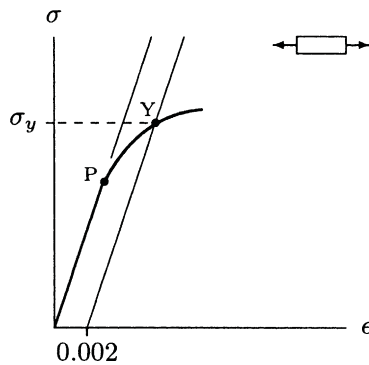

Fig. 13.13 Offset method

Note that a given material may behave differently under different load and environmental conditions. If the curve shown in Fig. 13.12 represents the stress–strain relationship for a material under tensile loading, there may be a similar but different curve representing the stress–strain relationship for the same material under compressive or shear loading. Also, temperature is known to alter the relationship between stress and strain. For a given material and fixed mode of

loading, different stress–strain diagrams may be obtained under different temperatures. Furthermore, the data collected in a particular tension test may depend on the rate at which the tension is applied on the specimen. Some of these factors affecting the relationship between stress and strain will be discussed later.

13.7 Elastic Deformations

Consider the partial stress–strain diagram shown in Fig. 13.14. Y is the yield point, and in this case, it also represents the proportionality and elastic limits. σ_y is the yield strength and ε_y is the corresponding strain. (The $\sigma - \varepsilon$ curve beyond the elastic limit is not shown.) The straight line in Fig. 13.14 represents the stress–strain relationship in the elastic region. *Elasticity* is defined as the ability of a material to resume its original (stress free) size and shape upon removal of applied loads. In other words, if a load is applied on a material such that the stress generated in the material is equal to or less than σ_y, then the deformations that took place in the material will be completely recovered once the applied loads are removed (the material is unloaded).
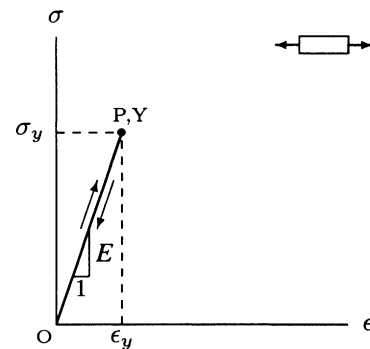
Fig. 13.14 Stress–strain diagram for a linearly elastic material. (↗: loading, ↙: unloading)

An elastic material whose $\sigma - \varepsilon$ diagram is a straight line is called a *linearly elastic* material. For such a material, the stress is linearly proportional to strain, and the constant of proportionality is called the *elastic* or *Young's modulus* of the material. Denoting the elastic modulus with E:

$$\sigma = E\varepsilon. \tag{13.5}$$

The elastic modulus, E, is equal to the slope of the $\sigma - \varepsilon$ diagram in the elastic region, which is constant for a linearly elastic material. E represents the *stiffness* of a material, such that the higher the elastic modulus, the stiffer the material.

The distinguishing factor in linearly elastic materials is their elastic moduli. That is, different linearly elastic materials have different elastic moduli. If the elastic

modulus of a material is known, then the mathematical definitions of stress and strain ($\sigma = F/A$ and $\varepsilon = \Delta l/l$) can be substituted into Eq. (13.5) to derive a relationship between the applied load and corresponding deformation:

$$\Delta l = \frac{F\,l}{E\,A}. \qquad (13.6)$$

In Eq. (13.6), F is the magnitude of the tensile or compressive force applied to the material, E is the elastic modulus of the material, A is the area of the surface that cuts the line of action of the applied force at right angles, l is the length of the material measured along the line of action of the applied force, and Δl is the amount of elongation or shortening in l due to the applied force. For a given linearly elastic material (or any material for which the deformations are within the linearly elastic region of the $\sigma - \varepsilon$ diagram) and applied load, Eq. (13.6) can be used to calculate the corresponding deformation. This equation can be used when the object is under a tensile or compressive force.

Not all elastic materials demonstrate linear behavior. As illustrated in Fig. 13.15, the stress–strain diagram of a material in the elastic region may be a straight line up to the proportionality limit followed by a curve. A curve implies varying slope and nonlinear behavior. Materials for which the $\sigma - \varepsilon$ curve in the elastic region is not a straight line are known as *nonlinear elastic materials*. For a nonlinear elastic material, there is not a single elastic modulus because the slope of the $\sigma - \varepsilon$ curve is not constant throughout the elastic region. Therefore, the stress–strain relationships for nonlinear materials take more complex forms. Note, however, that even nonlinear materials may have a linear elastic region in their $\sigma - \varepsilon$ diagrams at low stress levels (the region between points O and P in Fig. 13.15).
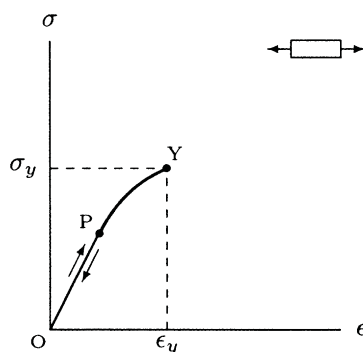
Fig. 13.15 Stress–strain diagram for a nonlinearly elastic material

Some materials may exhibit linearly elastic behavior when they are subjected to shear loading (Fig. 13.16). For such materials, the shear stress, τ, is linearly proportional to the shear strain, γ, and the constant of proportionality is
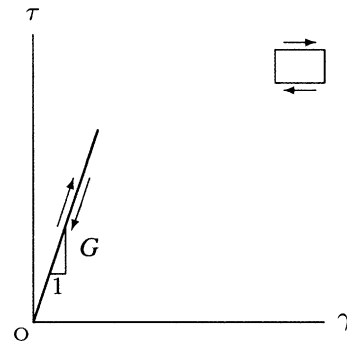
Fig. 13.16 Shear stress versus shear strain diagram for a linearly elastic material

called the *shear modulus* or the *modulus of rigidity*, which is commonly denoted with the symbol G:

$$\tau = G\,\gamma. \qquad (13.7)$$

The shear modulus of a given linear material is equal to the slope of the $\tau - \gamma$ curve in the elastic region. The higher the shear modulus, the more rigid the material.

Note that Eqs. (13.5) and (13.6) relate stresses to strains for linearly elastic materials and are called *material functions*. Obviously, for a given material, there may exist different material functions for different modes of deformation. There are also *constitutive equations* that incorporate all material functions.

13.8 Hooke's Law

The load-bearing characteristics of elastic materials are similar to those of springs, which was first noted by Robert Hooke. Like springs, elastic materials have the ability to store potential energy when they are subjected to externally applied loads. During unloading, it is the release of this energy that causes the material to resume its undeformed configuration. A linear spring subjected to a tensile load will elongate, the amount of elongation being linearly proportional to the applied load (Fig. 13.17). The constant of proportionality between the load and the deformation is usually denoted with the symbol k, which is called the *spring constant* or *stiffness* of the spring. For a linear spring with a spring constant k, the relationship between the applied load F and the amount of elongation d is:

$$F = kd. \qquad (13.8)$$

By comparing Eqs. (13.5) and (13.8), it can be observed that stress in an elastic material is analogous to the force applied to a spring, strain in an elastic material is analogous

to the amount of deformation of a spring, and the elastic modulus of an elastic material is analogous to the spring constant of a spring. This analogy between elastic materials and springs is known as *Hooke's law*.
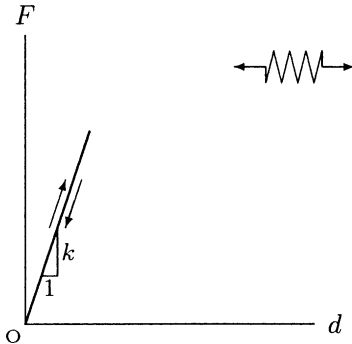

Fig. 13.17 Load-elongation diagram for a linear spring

13.9   Plastic Deformations

We have defined elasticity as the ability of a material to regain completely its original dimensions upon removal of the applied forces. Elastic behavior implies the absence of permanent deformation. On the other hand, *plasticity* implies permanent deformation. In general, materials undergo plastic deformations following elastic deformations when they are loaded beyond their elastic limits or yield points.

Consider the stress–strain diagram of a material shown in Fig. 13.18. Assume that a specimen made of the same material is subjected to a tensile load and the stress, σ, in the specimen is brought to such a level that $\sigma > \sigma_y$. The corresponding strain in the specimen is measured as ε. Upon removal of the applied load, the material will recover the plastic deformation that had taken place by following an unloading path parallel to the initial linearly elastic region (straight line between points O and P). The point where this
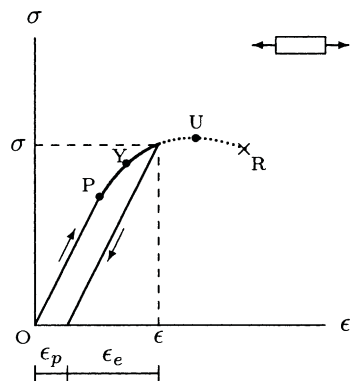

Fig. 13.18 Plastic deformation

path cuts the strain axis is called the plastic strain, ε_p, that signifies the extent of permanent (unrecoverable) shape change that has taken place in the specimen.

The difference in strains between when the specimen is loaded and unloaded ($\varepsilon - \varepsilon_p$) is equal to the amount of elastic strain, ε_e, that had taken place in the specimen and that was recovered upon unloading. Therefore, for a material loaded to a stress level beyond its elastic limit, the total strain is equal to the sum of the elastic and plastic strains:

$$\varepsilon = \varepsilon_e + \varepsilon_p. \qquad (13.9)$$

The elastic strain, ε_e, is completely recoverable upon unloading, whereas the plastic strain, ε_p, is a permanent residue of the deformations.

13.10   Necking

As defined in Sect. 13.6, the largest stress a material can endure is called the ultimate strength of that material. Once a material is subjected to a stress level equal to its ultimate strength, an increased rate of deformation can be observed, and in most cases, continued yielding can occur even by reducing the applied load. The material will eventually fail to hold any load and rupture. The stress at failure is called the rupture strength of the material, which may be lower than its ultimate strength. Although this may seem to be unrealistic, the reason is due to a phenomenon called *necking* and because of the manner in which stresses are calculated.

Stresses are usually calculated on the basis of the original cross-sectional area of the material. Such stresses are called *conventional stresses*. Calculating a stress by dividing the applied force with the original cross-sectional area is convenient but not necessarily accurate. The *true* or *actual stress* calculations must be made by taking the cross-sectional area of the deformed material into consideration. As illustrated in Fig. 13.19, under a tensile load, a material may elongate in the direction of the applied load but contract in the transverse directions. At stress levels close to the breaking point, the elongation may occur very rapidly and the material may narrow simultaneously. The cross-sectional area at the narrowed section decreases, and although the force required to further deform the material may decrease, the force per
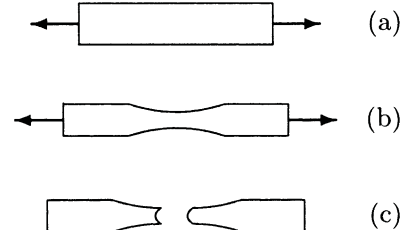

Fig. 13.19 Necking

unit area (stress) may increase. As illustrated with the dotted curve in Fig. 13.20, the actual stress–strain curve may continue having a positive slope, which indicates increasing strain with increasing stress rather than a negative slope, which implies increasing strain with decreasing stress. Also the rupture point and the point corresponding to the ultimate strength of the material may be the same.
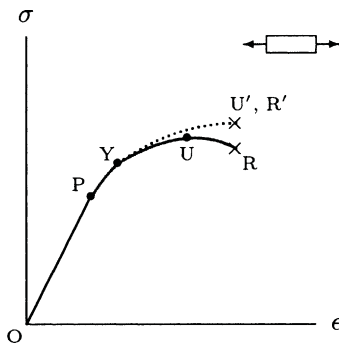
Fig. 13.20 Conventional (*solid curve*) and actual (*dotted curve*) stress–strain diagrams

13.11 Work and Strain Energy

In dynamics, *work done* is defined as the product of force and the distance traveled in the direction of applied force, and *energy* is the capacity of a system to do work. Stress and strain in deformable body mechanics are, respectively, related to force and displacement. Stress multiplied by area is equal to force, and strain multiplied by length is displacement. Therefore, the product of stress and strain is equal to the work done on a body per unit volume of that body, or the *internal work* done on the body by the externally applied forces. For an elastic body, this work is stored as an internal *elastic strain energy*, and it is the release of this energy that brings the body back to its original shape upon unloading. The maximum elastic strain energy (per unit volume) that can be stored in a body is equal to the total area under the $\sigma - \varepsilon$ diagram in the elastic region (Fig. 13.21). There is also a *plastic strain energy* that is dissipated as heat while deforming the body.
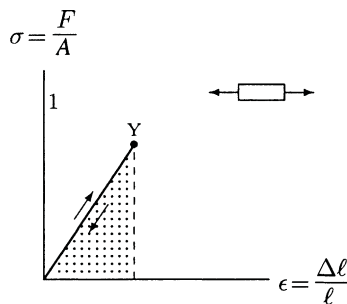
Fig. 13.21 Internal work done and elastic strain energy per unit volume

13.12 Strain Hardening

Figure 13.22 represents the $\sigma - \varepsilon$ diagram of a material. Assume that the material is subjected to a tensile force such that the stress generated is beyond the elastic limit (yield point) of the material. The stress level in the material is indicated with point A on the $\sigma - \varepsilon$ diagram. Upon removal of the applied force, the material will follow the path AB which is almost parallel to the initial, linear section OP of the $\sigma - \varepsilon$ diagram. The strain at B corresponds to the amount of plastic deformation in the material. If the material is reloaded, it will exhibit elastic behavior between B and A, the stress at A being the new yield strength of the material. This technique of changing the yield point of a material is called *strain hardening*. Since the stress at A is greater than the original yield strength of the material, strain hardening increases the yield strength of the material. Upon reloading, if the material is stressed beyond A, then the material will deform according to the original $\sigma - \varepsilon$ curve for the material.
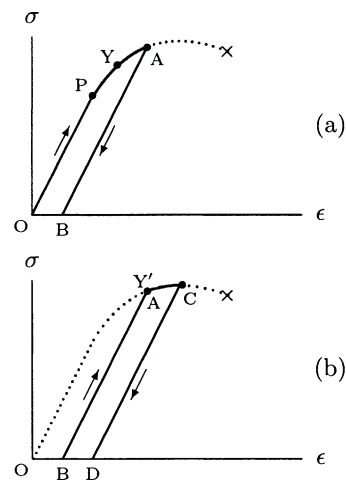
Fig. 13.22 Strain hardening

13.13 Hysteresis Loop

Consider the $\sigma - \varepsilon$ diagram shown in Fig. 13.23. Between points O and A, a tensile force is applied on the material and the material is deformed beyond its elastic limit. At A, the tensile force is removed, and the line AB represents the unloading path. At B, the material is reloaded, this time with a compressive force. At C, the compressive force applied on the material is removed. Between C and O, a second unloading occurs, and finally the material resumes its original shape. The loop OABCO is called the *hysteresis loop*, and the area enclosed by this loop is equal to the total strain energy dissipated as heat to deform the body in tension and compression.
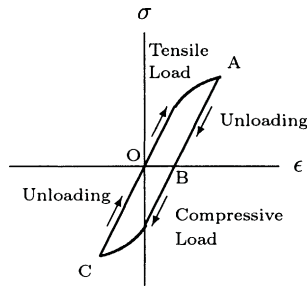
Fig. 13.23 Hysteresis loop

13.14 Properties Based on Stress–Strain Diagrams

As defined earlier, the elastic modulus of a material is equal to the slope of its stress–strain diagram in the elastic region. The elastic modulus is a relative measure of the *stiffness* of one material with respect to another. The higher the elastic modulus, the stiffer the material and the higher the resistance to deformation. For example, material 1 in Fig. 13.24 is stiffer than material 2.
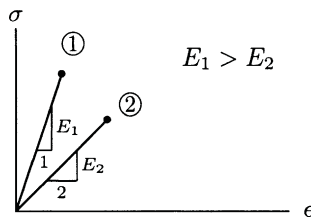

Fig. 13.24 Material 1 is stiffer than material 2

A *ductile* material is one that exhibits a large plastic deformation prior to failure. For example, material 1 in Fig. 13.25 is more ductile than material 2. A *brittle* material, on the other hand, shows a sudden failure (rupture) without undergoing a considerable plastic deformation. Glass is a typical example of a brittle material.
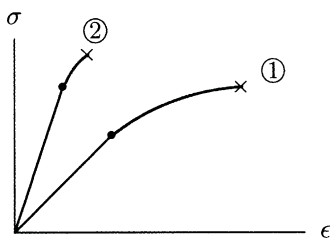

Fig. 13.25 Material 1 is more ductile and less brittle than 2

Toughness is a measure of the capacity of a material to sustain permanent deformation. The toughness of a material is measured by considering the total area under its stress–strain diagram. The larger this area, the tougher the material. For example, material 1 in Fig. 13.26 is tougher than material 2.
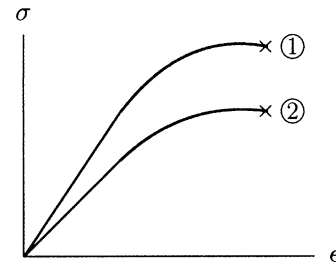

**Fig. 13.26** Material 1 is tougher than material 2

The ability of a material to store or absorb energy without permanent deformation is called the *resilience* of the material. The resilience of a material is measured by its *modulus of resilience* which is equal to the area under the stress–strain curve in the elastic region. The modulus of resilience is equal to $\sigma_y \varepsilon_y / 2$ or $\sigma_y^2 / 2E$ for linearly elastic materials.

Although they are not directly related to the stress–strain diagrams, there are other important concepts used to describe material properties. A material is called *homogeneous* if its properties do not vary from location to location within the material. A material is called *isotropic* if its properties are independent of direction or orientation. A material is called *incompressible* if it has a constant density.

13.15 Idealized Models of Material Behavior

Stress–strain diagrams are most useful when they are represented by mathematical functions. The stress–strain diagrams of materials may come in various forms, and it may not be possible to find a single mathematical function to represent them. For the sake of mathematical modeling and the analytical treatment of material behavior, these diagrams can be simplified. Some of these diagrams representing certain idealized material behavior are illustrated in Fig. 13.27.

A *rigid* material is one that cannot be deformed even under very large loads (Fig. 13.27a). A *linearly elastic* material is one for which the stress and strain are linearly proportional, with the modulus of elasticity being the constant of proportionality (Fig. 13.27b). A *rigid-perfectly plastic* material does not exhibit any elastic behavior, and once a critical stress level is reached, it will deform continuously and permanently until failure (Fig. 13.27c). After a linearly elastic response, a *linearly elastic-perfectly plastic* material is one that deforms continuously at a constant stress level (Fig. 13.27d). Figure 13.27e represents the stress–strain diagram for *rigid-linearly plastic* behavior.
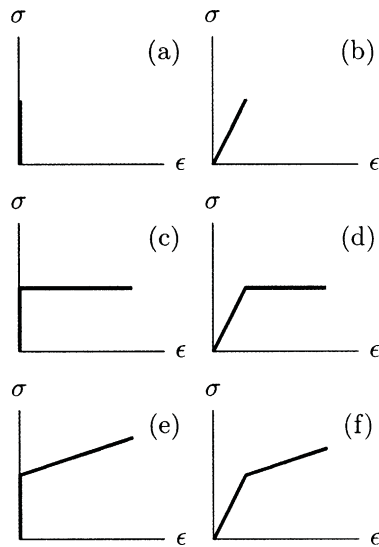
Fig. 13.27 Idealized models of material behavior

The stress–strain diagram of a *linearly elastic-linearly plastic* material has two distinct regions with two different slopes (bilinear), in which stresses and strains are linearly proportional (Fig. 13.27f).

13.16 Mechanical Properties of Materials

Table 13.2 lists properties of selected materials in terms of their tensile yield strengths (σ_y), tensile ultimate strengths (σ_u), elastic moduli (E), shear moduli (G), and Poisson's ratios (v). The significance of the Poisson's ratio will be discussed in the following chapter. Note that mechanical properties of a material can vary depending on many factors including its content (e.g., in the case of steel, its carbon content) and the processes used to manufacture the material (e.g., hard rolled, strain hardened, or heat treated). As will be discussed in later chapters, biological materials exhibit time-dependent properties. That is, their response to external forces depends on the rate at which the forces are applied. Bone, for example, is an anisotropic material. Its response to tensile loading in different directions is different, and

different elastic moduli are established to account for its response in different directions. Figure 13.28 illustrates the tensile stress-strain diagrams of a few materials. Steel and aluminum are relatively stiff and have high ultimate strengths. Glass and dry bone are brittle. Wet bone has the lowest ultimate strength and elastic modulus. Therefore, the values listed in Table 13.2 are some averages and ranges, and are aimed to provide a sense of the orders of magnitude of numbers involved.
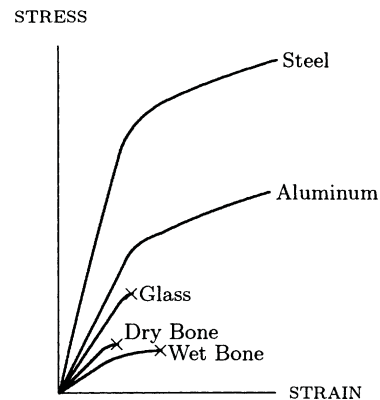
Fig. 13.28 Gross comparison of stress–strain diagrams of selected materials

13.17 Example Problems

The following examples will demonstrate some of the uses of the concepts introduced in this chapter.

Example 13.1 A circular cylindrical rod with radius $r = 1.26$ cm is tested in a uniaxial tension test (Fig. 13.29). Before applying a tensile force of $F = 1,000$ N, two points A and B that are at a distance $l_0 = 30$ cm (gage length) are marked on the rod. After the force is applied, the distance between A and B is measured as $l_1 = 31.5$ cm.

Determine the tensile strain and average tensile stress generated in the rod.

Table 13.2 Average mechanical properties of selected materials

Material	Yield strength σ_y (MPa)	Ultimate strength σ_u (MPa)	Elastic modulus E (GPa)	Shear modulus G (MPa)	Poisson's ratio v
Muscle	–	0.2	–	–	0.49
Tendon	–	70	0.4	–	0.40
Skin	–	8	0.5	–	0.49
Cortical bone	80	130	17	3.3	0.40
Glass	35–70	–	70–80	–	–
Cast iron	40–260	140–380	100–190	42–90	0.29
Aluminum	60–220	90–390	70	28	0.33
Steel	200–700	400–850	200	80	0.30
Titanium	400–800	500–900	100	45	0.34