BME 311
Biomechanics
Week 2
Moment and Torque

Textbook:
Basic Biomechanics of the Musculoskeletal System, 4th Ed., 2012, Edited by Margareta Nordin, Victor H. Frankel

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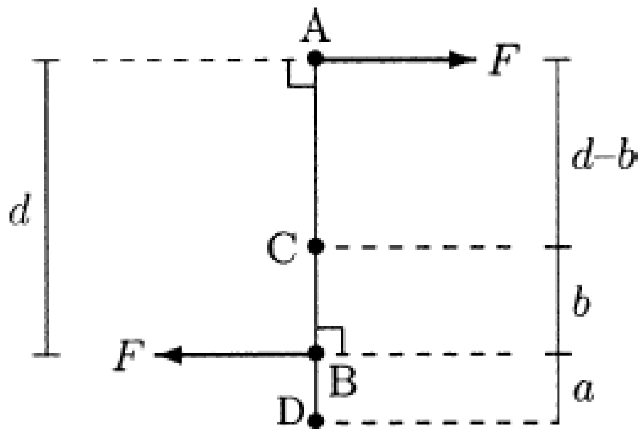

Fig. 3.21 A couple

What is the moment about points A, B, C, D?

Point A: dF (cw)
Point B: dF (cw)
Point C: (d-b)F+bF=dF (cw)
Point D: (d+a)F-aF=dF (cw)

- A special arrangement of forces is called **couple**, which is formed by two parallel forces with equal magnitude and opposite directions.
- The couple has the same moment about every point in space

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3.8 Translation of Forces

The overall effect of a pair of forces applied on a rigid body is zero if the forces have an equal magnitude and the same line of action, but are acting in opposite directions. Keeping this in mind, consider the force with magnitude F applied at point P_1 in Fig. 3.22a. As illustrated in Fig. 3.22b, this force may be translated to point P_2 by placing a pair of forces at P_2 with equal magnitude (F), having the same line of action, but acting in opposite directions. Note that the original force at point P_1 and the force at point P_2 that is acting in a direction opposite to that of the original force form a couple. This couple produces a counterclockwise moment with magnitude $M = dF$, where d is the shortest distance between the lines of action of forces at P_1 and P_2. Therefore, as illustrated in Fig. 3.22c, the couple can be replaced by the couple-moment.

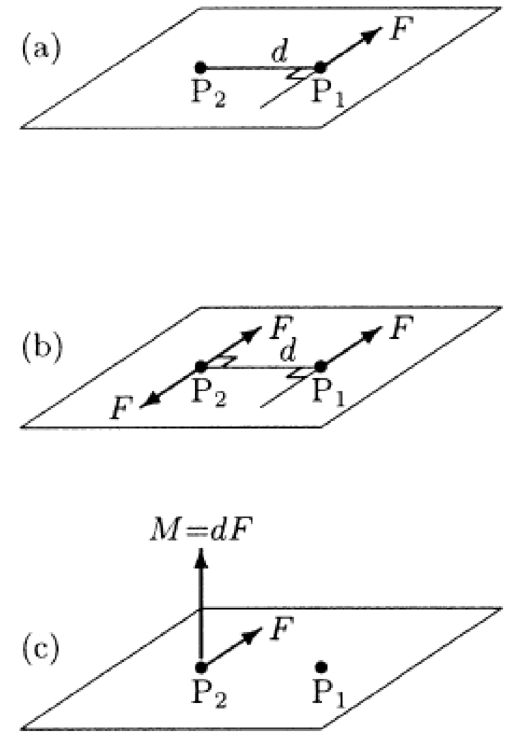

Fig. 3.22 Translation of a force from point P_1 to P_2

Provided that the original force was applied to a rigid body, the one-force system in Fig. 3.22a, the three-force system in Fig. 3.22b, and the one-force and one-couple-moment system in Fig. 3.22c are mechanically equivalent.

3.9 Moment as a Vector Product

Consider vectors $\underline{A}$ and $\underline{B}$, shown in Fig. 3.23. The cross product of $\underline{A}$ and $\underline{B}$ is equal to a third vector, $\underline{C}$:

$$\underline{C} = \underline{A} \times \underline{B} \tag{3.5}$$

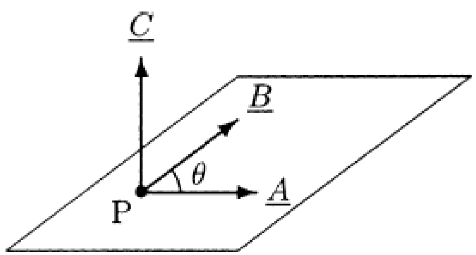

Fig. 3.23 $\underline{C}$ is the vector product of $\underline{A}$ and $\underline{B}$

The product vector $\underline{C}$ has the following properties:

- The magnitude of $\underline{C}$ is equal to the product of the magnitude of $\underline{A}$, the magnitude of $\underline{B}$, and $\sin\theta$, where θ is the smaller angle between $\underline{A}$ and $\underline{B}$.

$$C = A\ B\ \sin\ \theta \tag{3.6}$$

- The line of action of $\underline{C}$ is perpendicular to the plane formed by vectors $\underline{A}$ and $\underline{B}$.
- The direction and sense of $\underline{C}$ obeys the right-hand rule.

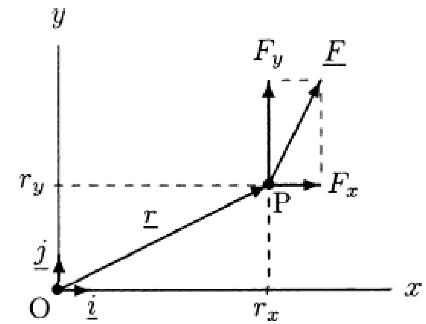

Fig. 3.24 The moment about O is $\underline{M} = \underline{r} \times \underline{F}$

and force vectors. The *position vector* of a point P with respect to another point O is defined by an arrow drawn from point O to point P. To help understand the definition of moment as a vector product, consider Fig. 3.24. Force $\underline{F}$ acts in the xy-plane and has a point of application at point P. Force $\underline{F}$ can be expressed in terms of its components F_x and F_y along the x and y directions:

$$\underline{F} = F_x\underline{i} + F_y\underline{j}. \tag{3.7}$$

The position vector of point P with respect to point O is represented by vector $\underline{r}$, which can be written in terms of its components:

$$\underline{r} = r_x\ \underline{i} + r_y\ \underline{j} \tag{3.8}$$

The moment of force $\underline{F}$ about point O is equal to the vector product of the position vector $\underline{r}$ and force vector $\underline{F}$:

$$\underline{M} = \underline{r} \times \underline{F}. \tag{3.9}$$

3.9 Moment as a Vector Product

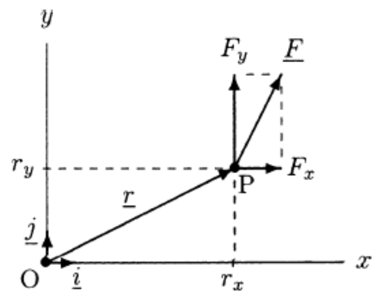

Fig. 3.24 The moment about O is $\underline{M} = \underline{r} \times \underline{F}$

Using (3.7) and (3.8), (3.9) can alternatively be written as:

$$\begin{aligned}\underline{M} &= (r_x\underline{i} + r_y\underline{j}) \times (F_x\underline{i} + F_y\underline{j}) \\ &= r_xF_x(\underline{i} \times \underline{i}) + r_xF_y(\underline{i} \times \underline{j}) \\ &+ r_yF_x(\underline{j} \times \underline{i}) + r_yF_y(\underline{j} \times \underline{j}).\end{aligned} \tag{3.10}$$

Recall that $\underline{i} \times \underline{j} = \underline{j} \times \underline{j} = 0$ since the angle that a unit vector makes with itself is zero, and $\sin 0° = 0$. $\underline{i} \times \underline{j} = \underline{k}$ because the angle between the positive x axis and the positive y axis is 90° ($\sin 90° = 1$). On the other hand, $\underline{j} \times \underline{i} = -\underline{k}$. For the last two cases, the product is either in the positive z (counterclockwise or out of the page) or negative z (clockwise or into the page) direction. z and unit vector $\underline{k}$ designate the direction perpendicular to the xy-plane (Fig. 3.25). Now, (3.10) can be simplified as:

$$\underline{M} = (r_xF_y - r_yF_x)\underline{k}. \tag{3.11}$$

Example:

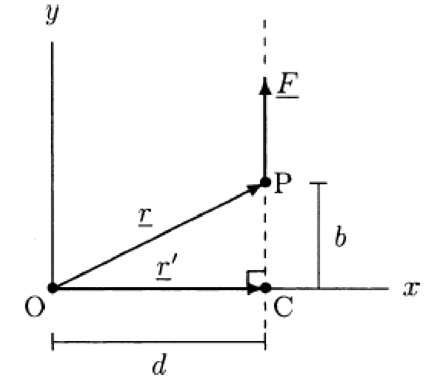

Fig. 3.26 $M = dF$ (ccw)

$$M = dF \quad \text{(ccw)}. \tag{3.12}$$

The force vector is acting in the positive y direction. Therefore,

$$\underline{F} = F\underline{j}. \tag{3.13}$$

If b is the y coordinate of the point of application of the force, then the position vector of point P is

$$\underline{r} = d\underline{i} + b\underline{j}. \tag{3.14}$$

Therefore, the moment of $\underline{F}$ about point O is

$$\begin{aligned}\underline{M} &= \underline{r} \times \underline{F} \\ &= (d\underline{i} + b\underline{j}) \times (F\underline{j}) \\ &= dF(\underline{i} \times \underline{j}) + bF(\underline{j} \times \underline{j}) \\ &= dF\underline{k}.\end{aligned} \tag{3.15}$$

Example 3.5 Figure 3.27a illustrates a person using an exercise machine. The "L"-shaped beam shown in Fig. 3.27b represents the left arm of the person. Points A and B correspond to the shoulder and elbow joints, respectively. Relative to the person, the upper arm (AB) is extended toward the left (x direction) and the lower arm (BC) is extended forward (z direction). At this instant, the person is holding a handle that is connected by a cable to a suspending weight. The weight applies an upward (in the y direction) force with magnitude F on the arm at point C. The lengths of the upper arm and lower arm are $a = 25$ cm and $b = 30$ cm, respectively, and the magnitude of the applied force is $F = 200$ N.

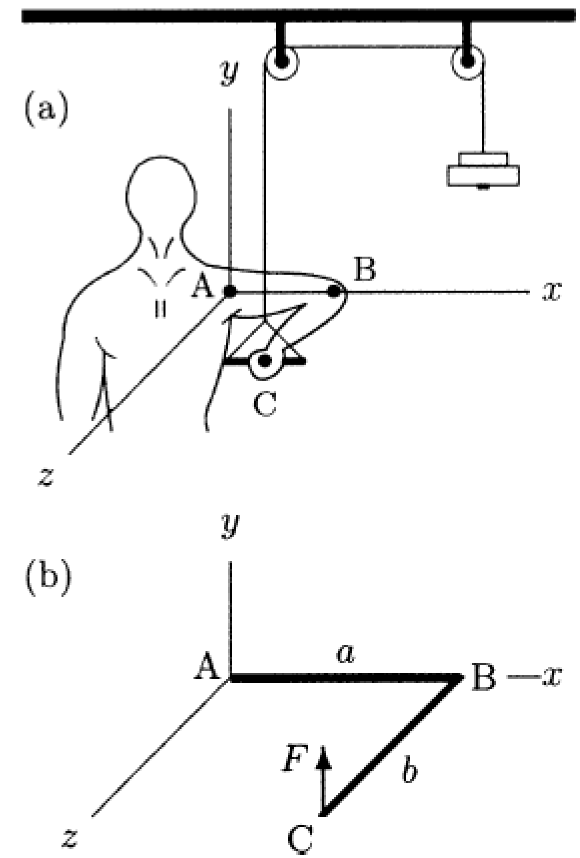

Fig. 3.27 Example 3.5

Explain how force F can be translated to the shoulder joint at point A, and determine the magnitudes and directions of moments developed at the lower and upper arms by $\underline{F}$.

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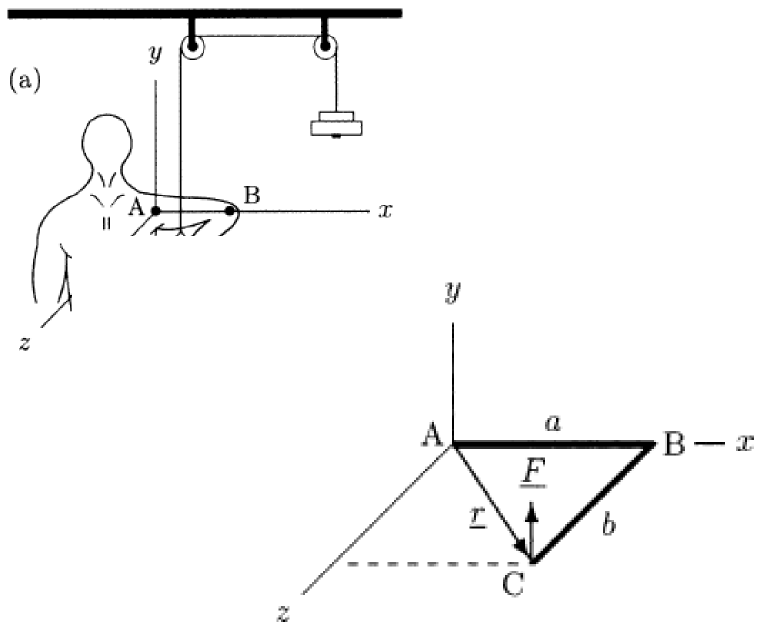

Fig. 3.29 Vector product method (Example 3.5)

Solution 2: Vector Product Method
The definition of moment as the vector product of the position and force vectors is more straightforward to apply. The position vector of point C (where the force is applied) with respect to point A (where the shoulder joint is located) and the force vector shown in Fig. 3.29 can be expressed as follows:

$$\underline{r} = a\underline{i} + b\underline{k}$$
$$\underline{F} = F\underline{j}.$$

The cross product of $\underline{r}$ and $\underline{F}$ will yield the moment of force $\underline{F}$ about point A:

$$\begin{aligned}
\underline{M} &= \underline{r} \times \underline{F} \\
&= (a\underline{i} + b\underline{k}) \times (F\underline{j}) \\
&= aF(\underline{i} \times \underline{j}) + bF(\underline{k} \times \underline{j}) \\
&= aF\underline{k} - bF\underline{i} \\
&= (0.25)(200)\underline{k} - (0.30)(200)\underline{i}. \\
&= 50k - 60i
\end{aligned}$$

The negative sign in front of $60\underline{i}$ indicates that the x component of the moment vector is acting in the negative x direction. Furthermore, there is no component associated with the unit vector $\underline{j}$, which implies that the y component of the moment vector is zero. Therefore, the moment about point A has the following components:

$$\begin{aligned}
M_x &= 60 \text{ N-m} \quad (-x \text{ direction}) \\
M_y &= 0 \\
M_z &= 50 \text{ N-m} \quad (+z \text{ direction}).
\end{aligned}$$

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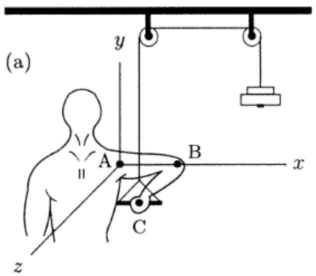

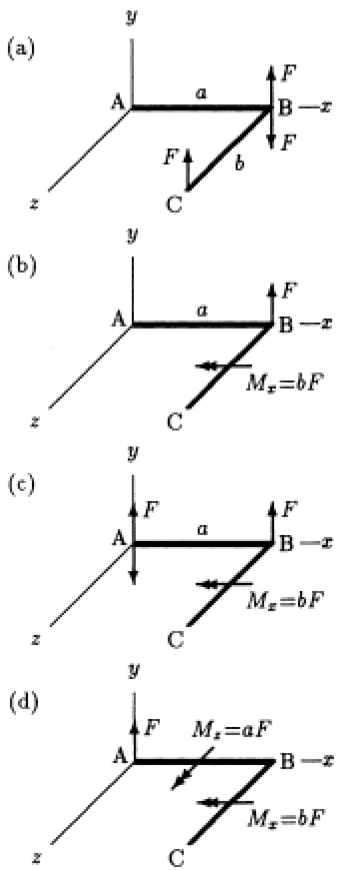

Fig. 3.28 Scalar method

Solution 1: Scalar Method

The scalar method of finding the moments generated on the lower arm (BC) and upper arm (AB) is illustrated in Fig. 3.28, and it utilizes the concepts of couple and couple-moment.

The first step is placing a pair of forces at point B with equal magnitude (F) and opposite directions, both being parallel to the original force at point C (Fig. 3.28a). The upward force at point C and the downward force at point B form a couple. Therefore, they can be replaced by a couple-moment (shown by a double-headed arrow in Fig. 3.28b). The magnitude of the couple-moment is bF. Applying the right-hand rule, we can see that the couple-moment acts in the negative x direction. If M_x refers to the magnitude of this couple-moment, then

$$M_x = bF \quad (-x \text{ direction}).$$

The next step is to place another pair of forces at point A where the shoulder joint is located (Fig. 3.28c). This time, the upward force at point B and the downward force at point A form a couple, and again, they can be replaced by a couple-moment (Fig. 3.28d). The magnitude of this couple-moment is aF, and it has a direction perpendicular to the xy-plane, or, it is acting in the positive z direction. Referring to the magnitude of this moment as M_z, then

$$M_z = aF \quad (+z \text{ direction}).$$

$$M_x = (0.30)(200) = 60 \text{ N-m}$$
$$M_z = (0.25)(200) = 50 \text{ N-m}.$$

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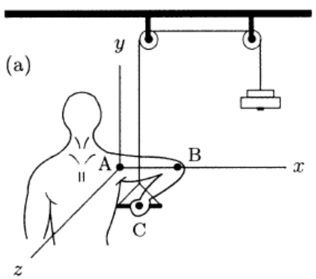

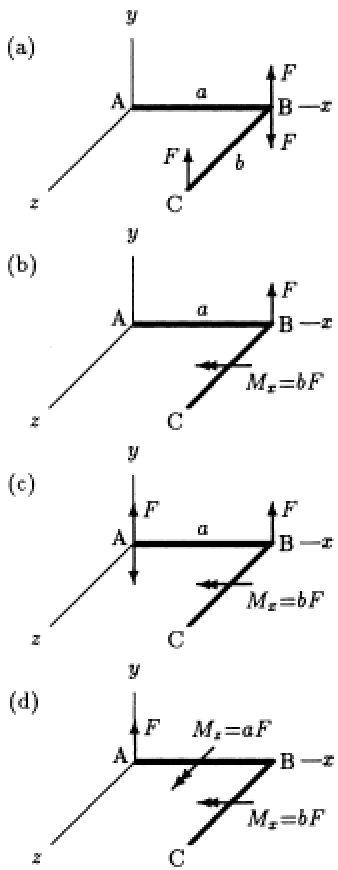

Fig. 3.28 Scalar method

The effect of the force applied at point C is such that at the elbow joint, the person feels an upward force with magnitude F and a moment with magnitude M_x that is trying to *rotate* the lower arm in the yz-plane. At the shoulder joint, the feeling is such that there is an upward force of F, a torque with magnitude M_x that is trying to twist the upper arm in the yz-plane, and a moment M_z that is trying to rotate or bend the upper arm in the xy-plane. If the person is able to hold the arm in this position, then he/she is producing sufficient muscle forces to counter-balance these applied forces and moments.